SECTION 15.3: PARTIAL DERIVATIVES

RECALL: The **derivative** of a function f, f' is $\dfrac{df}{dx} = f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, provided this limit exists.

If $f'(a)$ exists, then, geometrically, $f'(a)$ is the slope of the tangent line at $(a, f(a))$.

More generally, $f'(a)$ is the instantaneous rate of change of f with respect to x when $x = a$.

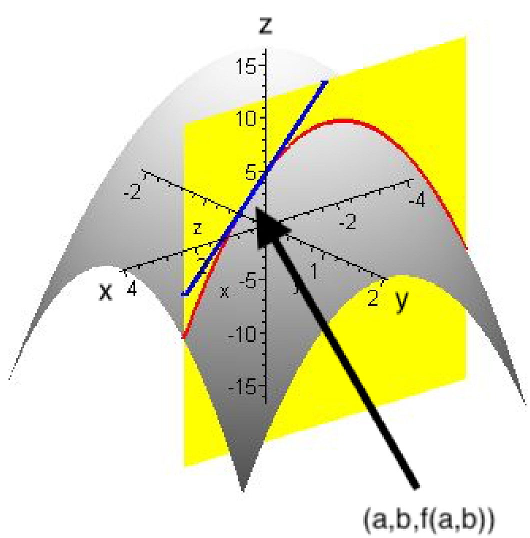
DEFINITION: Given a function $f(x, y)$:

- the **partial** derivative of f **with respect to** x is $\dfrac{\partial f}{\partial x} = f_x(x, y) = \lim\limits_{h \to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$
- the **partial** derivative of f **with respect to** y is $\dfrac{\partial f}{\partial y} = f_y(x, y) = \lim\limits_{k \to 0} \dfrac{f(x, y+k) - f(x, y)}{k}$

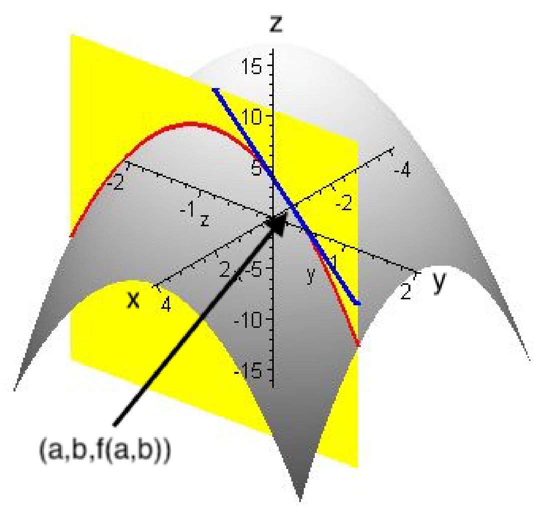
provided these limits exist.

Geometrically, $f_x(a, b)$ is the (z-x) slope of the tangent line of the curve $z = f(x, b)$ at the point $(a, b, f(a, b))$.

Geometrically, $f_y(a, b)$ is the (z-y) slope of the tangent line of the curve $z = f(a, y)$ at the point $(a, b, f(a, b))$.



Visualizing $f_x(a, b)$ as a slope



Visualizing $f_y(a, b)$ as a slope

More generally, $f_x(a, b)$ is the instantaneous rate of change of f with respect to x when $(x, y) = (a, b)$ and, likewise, $f_y(a, b)$ is the instantaneous rate of change of f with respect to y when $(x, y) = (a, b)$.

COMPUTATIONALLY:

To find $\dfrac{\partial f}{\partial x}$, take the derivative of f treating x as a variable and y as a constant.

To find $\dfrac{\partial f}{\partial y}$, take the derivative of f treating y as a variable and x as a constant.

EXAMPLE 1: Let $f(x, y) = 3x^2y - \cos(xy)$. Find and simplify:

1. $f_x(x, y)$

Ans: $f_x(x, y) = 6xy + y\sin(xy)$

2. $f_y(x, y)$

Ans: $f_y(x, y) = 3x^2 + x\sin(xy)$

NOTATION FOR HIGHER ORDER DERIVATIVES

- $f_{xx}(x, y) = \dfrac{\partial^2 f}{\partial x^2}$ means to take the derivative of f twice with respect to x.
- $f_{yy}(x, y) = \dfrac{\partial^2 f}{\partial y^2}$ means to take the derivative of f twice with respect to y.
- $f_{xy}(x, y) = \dfrac{\partial^2 f}{\partial y \partial x}$ means to take the derivative of f with respect to x then with respect to y.
- $f_{yx}(x, y) = \dfrac{\partial^2 f}{\partial x \partial y}$ means to take the derivative of f with respect to y then with respect to x.

NOTE: Think: $f_{xy}(x, y) = (f_x(x, y))_y$ or $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$

EXAMPLE 2: Let $f(x,y) = 3x^2y - \cos(xy)$. Find and simplify:

1. $f_{xx}(x,y)$

Ans: $f_{xx}(x,y) = 6y + y^2\cos(xy)$

2. $f_{yy}(x,y)$

Ans: $f_{yy}(x,y) = x^2\cos(xy)$

3. $f_{xy}(x,y)$

Ans: $f_{xy}(x,y) = 6x + \sin(xy) + xy\cos(xy)$

4. $f_{yx}(x,y)$

Ans: $f_{yx}(x,y) = 6x + \sin(xy) + xy\cos(xy)$

THEOREM: If f_{xy} and f_{yx} are continuous, then $f_{xy}(x,y) = f_{yx}(x,y)$.

EXAMPLE 3: One of the more common applications of the Calculus of Several Variables is in the Economics realm. The function below is called a 'Cobb-Douglas production function' and relates the production level Q (quantity of items produced) by a factory to the amount of money spent on labor x (salaries, benefits, etc.) and the amount of money spent on capital investments y (buildings, machinery, etc.)

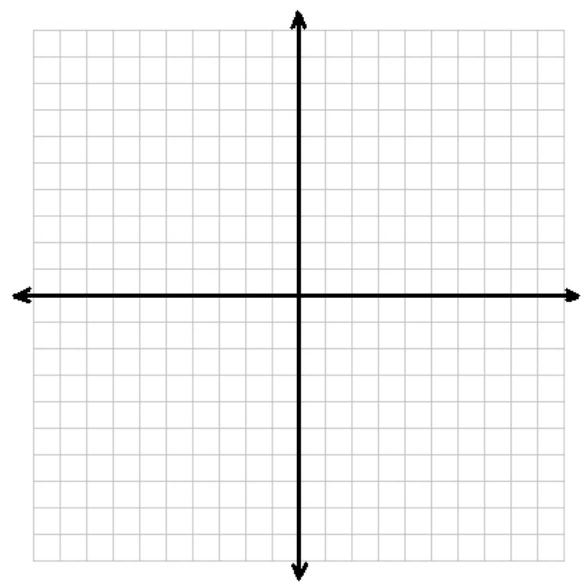
$$Q(x, y) = 150x^{1/3}y^{2/3}, \quad x > 0, \quad y > 0$$

Here Q is the amount of units produced annually, in *thousands*, and x and y represent the amount of money, in *millions of dollars*, spent annually on labor and capital, respectively. Currently, \$1 million is spent on labor and \$8 million is spent on capital.

1. How many items are currently being produced? (Remember, Q is being measured in *thousands*.)

Ans: $Q(1, 8) = 600$ so currently 600,000 are being produced.

2. Sketch the level curve $Q(x, y) = 600$. (This is called an 'isoquant.') Interpret what it means (in terms of labor, capital, and production) for a point (x, y) to lie on this level curve.



3. Find and simplify:

$Q_x(x, y)$

Ans: $Q_x(x, y) = 50x^{-2/3}y^{2/3}$

$Q_y(x, y)$

Ans: $Q_y(x, y) = 100x^{1/3}y^{-1/3}$

4. Find and simplify $Q_x(1, 8)$ and interpret what this number means in terms of a rate of change.

Ans: $Q_x(1, 8) = 200$.

For each additional \$1 million spent on labor, we can expect an increase in production of 200,000 units.

5. Find and simplify $Q_y(1, 8)$ and interpret what this number means in terms of a rate of change.

Ans: $Q_y(1, 8) = 50$.

For each additional \$1 million spent on capital, we can expect an increase in production of 50,000 units.

6. If an additional \$40 million is available for investment, use $Q_x(1, 8)$ to estimate what the production level would be if all these additional monies were spent on labor. Find the actual production level if all these additional monies were invested in labor and compare the two answers.

Ans: $Q_x(1, 8) = 200$, so we expect an increase of $(40)(200{,}000) = 8{,}000{,}000$ items.

So we predict a production of 8,600,000. $Q(41, 8) \approx 2069$ which corresponds to 2,069,000 items.

7. If an additional \$40 million is available for investment in the factory, use $Q_y(1, 8)$ to estimate what the production level would be if all these additional monies were spent on capital. Find the actual production level if all these additional monies were invested in capital and compare the two answers.

Ans: $Q_y(1, 8) = 50$, so we expect an increase of $(40)(50{,}000) = 2{,}000{,}000$ items.

So we predict a production of 2,600,000. $Q(1, 48) \approx 1981$ which corresponds to 1,981,000 items.

EXAMPLE 4: The Ideal Gas Law states that $PV = nRT$. In this equation, P denotes the pressure of the gas, V is the volume of the gas, T is the temperature of the gas, and n is the number moles (which gives the number of) gas molecules. The quantity R here is a constant.

1. Solve the Ideal Gas Law for P and find $\dfrac{\partial P}{\partial V}$.

Ans: $P = \dfrac{nRT}{V}$ so $\dfrac{\partial P}{\partial V} = -\dfrac{nRT}{V^2}$

2. Solve the Ideal Gas Law for V and find $\dfrac{\partial V}{\partial T}$.

Ans: $V = \dfrac{nRT}{P}$ so $\dfrac{\partial V}{\partial T} = \dfrac{nR}{P}$

3. Solve the Ideal Gas Law for T and find $\dfrac{\partial T}{\partial P}$.

Ans: $T = \dfrac{PV}{nR}$ so $\dfrac{\partial T}{\partial P} = \dfrac{V}{nR}$

4. Show $\left(\dfrac{\partial P}{\partial V}\right)\left(\dfrac{\partial V}{\partial T}\right)\left(\dfrac{\partial T}{\partial P}\right) = -1$.

HOMEWORK: Section 15.3: 3 - 79 every other odd.